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Operator Mixing and the AdS/CFT correspondence.

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Abstract: We provide a direct prescription for computing the mixing among gauge invariant operators in $\mathcal{N} = 4$ SYM. Our approach is based on the action of the superalgebra on the states of the theory and thus it can be also applied to resolve the mixing in the dual string description. As an example, we focus on the supermultiplet containing the BMN operators with two impurities. On the field theory side, we derive the leading planar quantum corrections to the naive expression of the highest weight state. Then we use the same prescription in the BMN limit of the $\mathrm{AdS}_5 \times S^5$ string theory and derive the form of the 2-impurity highest weight state. The string expression matches nicely the SYM result and provides a prediction for the mixing due to higher order quantum corrections in field theory.

Keywords: AdS-CFT Correspondence, Penrose limit and pp-wave background.

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Contents

1. Introduction

The operator mixing is an important aspect of any quantum field theory. In the $\mathcal{N} = 4$ Super Yang-Mills theory the mixing of gauge invariant operators is strictly connected to the superconformal properties of the theory. In fact, conformal symmetry puts strong constraints on the form of two and three point correlators. However, in concrete examples, these constraints are satisfied only if the appropriate form of the operator is used. Actually, as it has been extensively discussed in the literature [1 − 7], this observation provides a concrete way to resolve the mixing and obtain an explicit expression for the true eigenstates of the dilatation operator. One starts by computing correlators among a set of (classical) operators with the same naive scaling dimension and the same quantum numbers. Requiring that these correlators take the form dictated by conformal invariance implies a redefinition (mixing) of the original naive basis for the set of operators considered. The most common approach along these lines is to compute the 2-point functions among the states considered and then look for an orthonormal basis.

$\mathcal{N} = 4$ SYM is characterized by two dimensionless parameters, the rank of the gauge group N and the 't Hooft coupling $\lambda = g^2 N$. Thus, we have two types of operator mixing: that between single and multi trace operators governed by the $1/N$ expansion,[1] and the mixing due to planar quantum corrections which can be perturbatively computed as a series in $\sqrt{\lambda}$ (of course, in general, the two types of corrections can combine and give terms

[1] In the BPS sector the 2-point correlators do not receive quantum corrections and the conformal invariance yields less stringent constraints. In the context of the AdS/CFT correspondence, it is anyway important to find an orthonormal basis for the BPS operator, which gives rise to an interesting combinatorial problem, see [8, 9] and reference therein.

suppressed both by $\sqrt{\lambda}$ and $1/N$). With the discovery of integrable structures in $\mathcal{N} = 4$ SYM [23], it was shown that, actually, the problem of diagonalizing the planar dilatation operator is equivalent to finding the spectrum of the Hamiltonian of a quantum spin chain. Nowadays, the complete Hamiltonian is known up to one loop (order g^2) and Bethe Ansatz techniques had been used to all orders in perturbation theory for particular subsectors.

In this paper we present a different approach to the operator mixing issue which directly relies on the $\mathcal{N} = 4$ superalgebra. We start from the well known statement that each operator in a supermultiplet is annihilated by some of the (conformal) supercharges. For instance, a non-BPS highest weight state O is annihilated by all superconformal charges (S, $\bar{S}$) and none of the standard supersymmetries (Q, $\bar{Q}$). Schematically we have

$$[S, O\,(x = 0)] = 0 \ , \quad \text{and} \qquad [Q, O\,(x = 0)] \neq 0 \ . \tag{1.1}$$

At the classical level, one can easily implement this requirement by using the standard variations of the elementary fields composing the gauge invariant operator. The easiest way to promote this approach to the quantum level is to study the Ward identities of the supersymmetric currents. For instance, the first equation in (1.1) can be rewritten as

$$\frac{\partial}{\partial y^\mu} \langle S^\mu(y)\, O_1(x_1)\, O(0)\rangle = -\mathrm{i}\,\langle \delta_S O_1(x_1)\, O(0)\rangle\, \delta^4(y - x_1)\,, \tag{1.2}$$

where S^μ is the current related to the conformal supercharge S, O_1 is an arbitrary operator and $[S, O_1] = \hat{O}_1$. The fact that O is annihilated by S translates into the absence in (1.2) of a term proportional to $\delta(y)$. Of course (1.2) must hold also beyond the tree-level approximation and we show that the explicit computation of the quantum corrections can be used to resolve the operator mixing.

In principle this approach can be applied to compute both the planar and the $1/N$ corrections to the naive form of the non-BPS gauge invariant operators. However, in our explicit example we focus only on the planar mixing; in particular we derive the leading quantum corrections to the highest weight state of the supermultiplet studied in [10]. This state is a generalization of the usual Konishi operator and has classical conformal dimension $J + 2$ transforming in the $[0, J, 0]$ representation of the SU(4) R-symmetry. As it was suggested in [7], we find that, at order $\sqrt{\lambda}$, the true highest weight state is a combination of the naive form containing only scalar fields and a correction term with two fermion impurities. This is reminiscent of the mixing discussed in [11] in the context of instanton corrections, but is new from the quantum spin chain approach. As we will discuss at the end of section 3, this mixing between scalar and fermion impurities is not captured by the one-loop Hamiltonian, even if it appears at order $\sqrt{\lambda}$. The situation is similar to the single/double trace mixing discussed in [4, 5]: the form of the operators at a certain order in perturbation theory requires the knowledge of the dilatation operator at higher orders.

A nice feature of the method discussed here is that it can also be applied directly on the string side of the AdS/CFT correspondence, thus shedding some light on how the operator mixing is realized in the dual string description. There is a one-to-one map between the field and the string theory superalgebra [12] and so we can look for the string states that are annihilated by the same supercharges used in the field theory computation.

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This approach provides a natural dictionary between the spectra of the two descriptions. Of course, its implementation in the full type IIB superstring theory on $\mathrm{AdS}_5 \times S^5$ is beyond reach right now. However, we can easily carry out this computation in the BMN limit [13], where we focus only on states with a large U(1) R-charge J. In this limit, type IIB string theory in the light cone is described by a free 2-dimensional world-sheet Lagrangian and the expression of all supercharges is explicitly known [14, 15]. We again consider the sector of 2-impurity states and derive the highest weight state of the multiplet in the free PP-wave string theory. The string expression matches the large J limit of the perturbative SYM result and provides a prediction for the higher order quantum corrections in field theory. However, the numerical agreement between the string and the field theory results is somehow surprising. BMN string theory is a reliable approximation of the full $AdS_5 \times S^5$ theory in the limit $\lambda, J \to \infty$, with $\frac{\lambda}{J^2} = \lambda'$ fixed, which is far away from the standard perturbative field theory ($\lambda \ll 1$). We extend the string result at weak coupling by supposing the validity of the BMN scaling. This is known to break down at four loops in gauge theory [16]. Thus, it is unlikely that this numerical agreement between the large J limit of the form of the operator and the small λ behavior of the string expression survives at higher loops.

The structure of the paper is the following. In section 2 we focus on free IIB string theory in the BMN limit. We summarize how the superalgebra is realized and we use it to build the highest weight states with two impurities. In the strong curvature limit, one recovers the usual expression which contains only scalar impurities and is directly connected to free SYM expression for the primary operators. The exact expression for the string highest weight states involves also fermion and scalar impurities suggesting a precise mixing pattern on the SYM side of the correspondence. In section 3 we focus on the SYM side of the correspondence. By studying the supersymmetric Ward identities we give a perturbative derivation of the action of the supercharges up to order g on gauge invariant operators with scalar fields. In [17] this result was derived by using exclusively the SU(2|3) subgroup of the $\mathcal{N} = 4$ symmetry algebra, together with just one dynamical input such as the known anomalous dimension of the Konishi operator. The action of the supersymmetry generators has been studied [18, 19] also in the plane-wave matrix model in a spirit similar to what is done in this paper. In section 4 we compare the strong curvature expansion of the two impurity string states derived in section 2 against the 1-loop corrected field theory highest weight states in the large J limit. The two results agree by using the standard BMN/SYM dictionary and this suggests that at order g^2 the field theory primary operators should contain also space-time derivatives. In the Conclusions we discuss some possible developments where the results presented in this paper can play an important role. Two appendices contain our conventions for the $\mathcal{N} = 4$ SYM theory and all technical results useful for the derivation discussed in section 3.

2. Operator mixing in (BMN) string theory

Let us consider type IIB string theory on the maximal supersymmetric PP-wave background [20]. In the light-cone gauge this theory is described by a free two dimensional

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action. So upon quantization we have eight towers of bosonic and fermionic harmonic oscillators ($a_n^\dagger$, $b_n^\dagger$) transforming in the vector and spinor representation respectively of the $\mathrm{SO}(4)\times\mathrm{SO}(4)$ subgroup of the full $\mathrm{SO}(2,4)\times\mathrm{SO}(6)$ isometry group of $\mathrm{AdS}_5\times S^5$ [14, 15]. The physical spectrum is a subset of the Fock space generated by these creations operators which consists of all states satisfying the level matching condition[2]

$$T|s\rangle = 0 \ , \qquad \text{with} \quad T=\sum_{n=1}^{\infty} n\left[\left(b_n^\dagger b_{-n}+b_{-n}^\dagger b_n\right)-\mathrm{i}\left(a_n^\dagger a_{-n}-a_{-n}^\dagger a_n\right)\right] , \tag{2.1}$$

where we suppressed all space-time indices that are contracted in the standard $\mathrm{SO}(4)\times\mathrm{SO}(4)$ invariant way. The light-cone Hamiltonian is

$$H=\frac{1}{\mu\alpha'|p^+|}\sum_{n=-\infty}^{\infty}\omega_n\left[a_n^\dagger a_n+b_n^\dagger b_n\right] , \tag{2.2}$$

where $\omega_n = \sqrt{n^2+(\mu\alpha' p^+)^2}$, p^+ is the light cone momentum of the state and μ is the parameter setting the curvature of the PP-wave background (in the following we will define, as usual, $\alpha\equiv\alpha' p^+$). This background preserves 32 supercharges. Half of them are purely kinematical and contain only the zero-mode oscillators

$$Q^+=\sqrt{2|\alpha|}\left[e(\alpha)\mathbb{P}^- b_0+\mathbb{P}^+ b_0^\dagger\right] , \quad \bar{Q}^+=\sqrt{2|\alpha|}\left[\mathbb{P}^- b_0^\dagger+e(\alpha)\mathbb{P}^+ b_0\right] , \tag{2.3}$$

where $\mathbb{P}^\pm=(1\pm\Pi)/2$ and Π is the appropriate 16×16 block of the matrix $\prod_{i'=1}^4\Gamma^{i'}$, where the index i' is restricted to the flavor $\mathrm{SO}(4)\subset\mathrm{SO}(6)$ and the Γ's (γ's) indicate the $\mathrm{SO}(1,9)$ ($\mathrm{SO}(8)$) Gamma matrices respectively. The remaining sixteen supercharges are dynamical and display a non-trivial dependence on $\mu\alpha$

$$\begin{aligned} Q^- &= e(\alpha)\sqrt{\frac{1}{2}}\gamma\left[a_0\left(1+e(\alpha)\Pi\right)b_0^\dagger+e(\alpha)a_0^\dagger\left(1-e(\alpha)\Pi\right)b_0\right]+ \\ &\quad +\frac{1}{\sqrt{|\alpha|}}\sum_{n=1}^{\infty}\sqrt{n}\gamma\left[a_n^\dagger P_n b_{-n}+e(\alpha)a_n P_n^{-1}b_n^\dagger+\mathrm{i}a_{-n}^\dagger P_n b_n-\mathrm{i}e(\alpha)a_{-n}P_n^{-1}b_{-n}^\dagger\right] , \\ \bar{Q}^- &= \sqrt{\frac{1}{2}}\gamma\left[e(\alpha)a_0\left(1-e(\alpha)\Pi\right)b_0^\dagger+a_0^\dagger\left(1+e(\alpha)\Pi\right)b_0\right]+ \\ &\quad +\frac{1}{\sqrt{|\alpha|}}\sum_{n=1}^{\infty}\sqrt{n}\gamma\left[a_n^\dagger P_n^{-1}b_n+e(\alpha)a_n P_n b_{-n}^\dagger+\mathrm{i}a_{-n}^\dagger P_n^{-1}b_{-n}-\mathrm{i}e(\alpha)a_{-n}P_n b_n^\dagger\right] , \end{aligned} \tag{2.4}$$

where $e(\alpha)=1$ if $\alpha>0$ while $e(\alpha)=-1$ if $\alpha<0$ and

$$\rho_n=\frac{\omega_n-n}{\mu\alpha} \ , \quad P_n^{\pm1}=\frac{1}{\sqrt{1-\rho_n^2}}(1\mp\rho_n\Pi) \ . \tag{2.5}$$

The standard prescription [13] for building the dictionary between string and field theory states is to identify the creation modes with the presence of "impurities" (i.e. fields

[2]We follow the notations of [21], with some small changes so to have a string theory supersymmetry algebra that agrees with the field theory one following from the conventions of appendix A.

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with $\Delta - J = 1$) in the corresponding gauge theory operator. Thus the dictionary is usually set-up at the level of the basic constituents (letters) by checking that they transform in the same way under the $\mathrm{SO}(4) \times \mathrm{SO}(4)$ symmetry of the problem. For instance, the relation between the gauge and string theory expression for the $\mathrm{SO}(4) \times \mathrm{SO}(4)$ singlet with two scalar impurities is usually written as follows[3]

$$\sum_{p=0}^{J} \cos\frac{\pi n(2p+3)}{J+3} \mathrm{Tr}\big[\Phi_{AB} Z^p \Phi^{AB} Z^{J-p}\big] \longleftrightarrow \sum_{i'=1}^{4} \alpha^{\dagger}{}^{i'}_{n} \alpha^{\dagger}{}^{i'}_{-n} |\alpha\rangle\,, \tag{2.6}$$

where $|\alpha\rangle$ is the vacuum state of fixed light-cone momentum p^+ and $\alpha_n^\dagger$ ($\alpha_{-n}^\dagger$) are the oscillators creating left (right) moving excitations on the string.[4]

Here we will follow a different approach to the construction of the field/string theory dictionary. We first identify the supersymmetry generators in the two descriptions by requiring that they satisfy the same algebra. Then, we derive the highest weight states of the string and field theory algebra separately. The first entry of the dictionary between the two spectra just consists in relating the two highest weight states. Then it is straightforward to build the dictionary for the whole supermultiplet: we just need to act on the the highest weight state in each description with supercharges that have been already identified. The two approaches yield the same dictionary between the string and the field theory spectra in the large $\mu\alpha$ limit. What is more surprising is that even the first subleading corrections agree, as we will see in section 4.

By comparing the string and field theory superalgebras, one obtains (for $\alpha > 0$) the following correspondence

$$Q_{\alpha,A=1,2} \leftrightarrow \mathbb{P}^+ Q^+\,, \quad Q_{\alpha,A=3,4} \leftrightarrow \mathbb{P}^+ Q^-\,, \quad \bar{Q}^{\dot\alpha,A=1,2} \leftrightarrow \mathbb{P}^- \bar{Q}^-\,, \quad \bar{Q}^{\dot\alpha,A=3,4} \leftrightarrow \mathbb{P}^- \bar{Q}^+\,, \tag{2.7}$$

where Q_α and $\bar{Q}^{\dot\alpha}$ are the standard gauge theory supercharges (A.15), and $Q^\pm$, $\bar{Q}^\pm$ are the supersymmetry operators in the PP-wave string theory (2.3)–(2.4). In the BPS sector a highest weight state is annihilated also by half of the transformations in (2.7). In fact, on the field theory side the operator $\mathrm{Tr}\big[Z^J\big]$ is invariant under transformation generated by $Q_{3,4}$ and $\bar{Q}^{1,2}$ and the same is true for the corresponding string state $|\alpha\rangle$. The other sixteen string supercharges correspond to the superconformal symmetries of the gauge theory description, see (A.17)

$$S_\alpha^{A=1,2} \leftrightarrow \mathbb{P}^+ \bar{Q}^+\,, \quad S_\alpha^{A=3,4} \leftrightarrow \mathbb{P}^+ \bar{Q}^-\,, \quad \bar{S}^{\dot\alpha}_{A=1,2} \leftrightarrow \mathbb{P}^- Q^-\,, \quad \bar{S}^{\dot\alpha}_{A=3,4} \leftrightarrow \mathbb{P}^- Q^+\,. \tag{2.8}$$

Thus any highest weight states should be annihilated by all operators in (2.8)

$$\mathbb{P}^+ \bar{Q}^+ |hws\rangle = \mathbb{P}^+ \bar{Q}^- |hws\rangle = \mathbb{P}^- Q^- |hws\rangle = \mathbb{P}^- Q^+ |hws\rangle = 0\,. \tag{2.9}$$

Then it is clear that we should focus on the string states that do not contain any $b_0^\dagger$ so that they are annihilated by $\mathbb{P}^+ \bar{Q}^+$ and $\mathbb{P}^- Q^+$. The conditions following from the remaining

[3] See appendix A for our field theory conventions.

[4] In all other formulae of this paper we use the string field theory oscillators a_n, which are related to the $\alpha_{\pm n}$ as follows: $\alpha_{\pm n} = \frac{1}{\sqrt{2}}(a_n \mp \mathrm{i} a_{-n})$.

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supercharges must be solved case by case: here we will consider the multiplets containing the states with two string creation operators and show that the 2-impurity highest weight states are not given simply by the eq. (2.6).

The first observation is that the string state in (2.6) is annihilated by the dynamical supercharges in (2.8) *only* in the $\mu\alpha \to \infty$ limit. In fact, when we compute the $\mathbb{P}^{\pm}$ projections of the dynamical supercharges in (2.8), we have to separate the terms with a Gamma matrix in the "flavor" SO(4) (indicated with an index i') from those with a Gamma in the "space-time" SO(4) (indicated with i). Π commutes with $\gamma^{i'}$ and anticommutes with γ^{i}. Thus, for instance, in the case $\alpha > 0$ we have that the string charges corresponding to $S_{\alpha}^{A=3,4}$ are

$$\begin{aligned}\mathbb{P}^{+}\bar{Q}^{-} = \sqrt{2}\left[\gamma^{i}a_0^{i}\mathbb{P}^{-}b_0^{\dagger} + \gamma^{i'}a_0^{i'\,\dagger}\mathbb{P}^{+}b_0\right] + \frac{1}{\sqrt{\mu|\alpha|}}\sum_{n=1}^{\infty}\sqrt{n}\Big\{ \\ \gamma^{i'}\left[a_n^{i'\dagger}\mathbb{P}^{+}b_n + \mathrm{i}a_{-n}^{i'\dagger}\mathbb{P}^{+}b_{-n}\right]U_n^{-\frac{1}{2}} + \gamma^{i'}\left[a_n^{i'}\mathbb{P}^{+}b_{-n}^{\dagger} - \mathrm{i}a_{-n}^{i'}\mathbb{P}^{+}b_n^{\dagger}\right]U_n^{\frac{1}{2}} \\ +\gamma^{i}\left[a_n^{i}\mathbb{P}^{-}b_{-n}^{\dagger} - \mathrm{i}a_{-n}^{i}\mathbb{P}^{-}b_n^{\dagger}\right]U_n^{-\frac{1}{2}} + \gamma^{i}\left[a_n^{i\dagger}\mathbb{P}^{-}b_n + \mathrm{i}a_{-n}^{i\dagger}\mathbb{P}^{-}b_{-n}\right]U_n^{\frac{1}{2}}\Big\},\end{aligned} \tag{2.10}$$

where $U_n^{\pm 1} \equiv \frac{1\mp\rho_{n(1)}}{1\pm\rho_{n(1)}}$ and the repeated indices are summed. It is interesting to consider the form of the dynamical supercharges in the large $\mu\alpha$ limit. In this limit we have that $U_n \sim n/(2\mu\alpha)$, so the terms with $U_n^{-1/2}$ dominate over those with $U_n^{1/2}$ and the leading contribution to (2.10) is schematically $\mathbb{P}^{+}\bar{Q}^{-} \sim \gamma^{i}a^{i}\mathbb{P}^{-}b^{\dagger}$ along the space-time directions and $\mathbb{P}^{+}\bar{Q}^{-} \sim \gamma^{i'}a^{i'\,\dagger}\mathbb{P}^{+}b$ along the flavor directions. This result can be matched directly against the $g \to 0$ form of the field theory superconformal transformation of appendix A: $S\psi(0) \sim Z(0)$ and $S\partial Z(0) \sim \sigma\psi(0)$, where again we have suppressed all numerical factors and indices. In the same way, the large $\mu\alpha$ limit of $\mathbb{P}^{+}Q^{-}$ and $\mathbb{P}^{-}\bar{Q}^{-}$ agrees with the gauge theory supersymmetry transformations at $g = 0$, as summarized by the dictionary (2.7).

Let us now consider the action of (2.10) on the string state in (2.6). It is clear that the second term of the second line annihilates this state only in the large $\mu\alpha$ limit, showing that it is not an exact highest weight state of the string superalgebra. This suggests that also on the field theory side the operator in (2.6) is a superconformal primary only in the $g_{YM} \to 0$ limit. On the string side it is clear how to modify the state in (2.6) so to find the true highest state weight of the multiplet. We need to add a contribution which contains two fermionic oscillators and is not annihilated by the first term of the second line in (2.10). The coefficient is chosen to satisfy (2.9). By repeating the same procedure also for $\mathbb{P}^{-}Q^{-}$, one obtains that the 2-impurity states satisfying (2.9) exactly are

$$|n\rangle = \frac{1}{4(1+U_n^2)}\left[a^{\dagger i'}_{\ n}a^{\dagger i'}_{\ n} + a^{\dagger i'}_{\ -n}a^{\dagger i'}_{\ -n} + 2U_n b_{-n}^{\dagger}\,\Pi\, b_n^{\dagger} - U_n^2\left(a^{\dagger i}_{\ n}a^{\dagger i}_{\ n} + a^{\dagger i}_{\ -n}a^{\dagger i}_{\ -n}\right)\right]|\alpha\rangle, \tag{2.11}$$

where the overall normalization has been fixed in such a way that the state is normalized to one: $\langle n|n\rangle = 1$. The main feature of (2.11) is the mixing between various types of impurities. At leading order in the $\mu\alpha \to \infty$ expansion, we have only scalar impurities ($a^{\dagger i'}_{\ n}$). The first corrections appear at order $\mathcal{O}(1/(\mu\alpha))$ and are quadratic in the fermionic oscillators. According to the standard dictionary between the PP-wave and the field theory parameters, this translates into a quantum correction of order λ'. At the next order ($\mathcal{O}(1/(\mu\alpha)^2)$) also vector impurities appear and we expect that the same pattern is present also in field theory. Finally, let us stress again that, starting from the state in (2.11), it is tedious but straightforward to build the whole supermultiplet by using the supercharges in (2.7).

3. Operator mixing in field theory

The analysis of the previous section has shown that in general the string theory highest weight states involve mixing between different kinds of impurities. It would be desirable to see the same pattern appearing in perturbative field theory. In this section, we evaluate the first correction to the classical form of the highest weight operator involving two impurities and find perfect agreement, in the appropriate limit, with the string theory expression (2.11). We follow the same approach discussed in the string theory context and compute the form of the highest weight state by looking for field theory operators that satisfy (1.1).

As already mentioned in the previous section, the gauge theory supersymmetry transformations at $g = 0$ agree with $\mu\alpha \to \infty$ limit of the superstring ones and the scalars are annihilated by all S and $\bar{S}$ (A.17). Thus any composite operator built solely from scalars is a primary state at leading order, and the 2-impurity SYM primaries are the operators already introduced in (2.6):

$$\mathcal{O}_n^{(0)J} = \sqrt{\frac{N_0^{-J-2}}{(J+3)}} \sum_{i=1}^{3} \sum_{p=0}^{J} \cos\frac{\pi n(2p+3)}{J+3} \mathrm{Tr}\big[Z_i Z^p \bar{Z}_i Z^{J-p}\big] \,, \tag{3.1}$$

where the normalization $N_0 = N/(8\pi^2)$ is fixed to have[5] $\langle \bar{\mathcal{O}}_n^{(0)J}(x)\, \mathcal{O}_n^{(0)J}(0)\rangle = \frac{(-1)^{J+2}}{(x^2)^{J+2}}$.

Things change if one considers the full interacting quantum theory. In this case, most of the aforementioned states are not annihilated by all the superconformal charges and the true primaries are not built with scalar impurities only. For instance, at first order in g, we have

$$\bar{S}^{\dot\alpha}_A \Phi_{BC}\Phi_{DE}(0) = -\mathrm{i}\,\frac{gN}{32\pi^2}\left(\epsilon_{ABC[D}\bar{\psi}_{E]}(0) - \epsilon_{ADE[B}\bar{\psi}^{\dot\alpha}_{C]}(0)\right) , \tag{3.2}$$

where $\epsilon_{ABC[D}\bar{\psi}_{E]} = \frac{1}{2}(\epsilon_{ABCD}\bar{\psi}_E - \epsilon_{ABCE}\bar{\psi}_D)$. If we restrict the indices to the SU(2|3) sector, this expression agrees[6] with that of [17]. Here we give a diagrammatic derivation of this result that immediately leads to the SU(2, 2|4) form of (3.2). The relevant field theory diagrams are depicted in figures 1 and 2, where the classical form of the superconformal transformation is combined with a Yukawa coupling.

For sake of concreteness, let us focus on the action of $\bar{S}_1$ on the scalar fields $Z\bar{Z}_1$; we compute the 3-point function

$$(G_3^{\mu})^{i\,k}_{j\,l} = \langle \bar{S}_1^{\mu\dot\alpha}(y)\ \big(\psi^3_\gamma\big)^i_j(x)\ \big(Z\bar{Z}_1\big)^k_l(0)\rangle \tag{3.3}$$

[5]The factor of $(-1)^{J+2}$ disappears after rotating to Euclidean spacetime.

[6]One has to take into account that our supersymmetry algebra agrees with that of [22], where the (super)conformal generators S and K are normalized in a different way from [17].

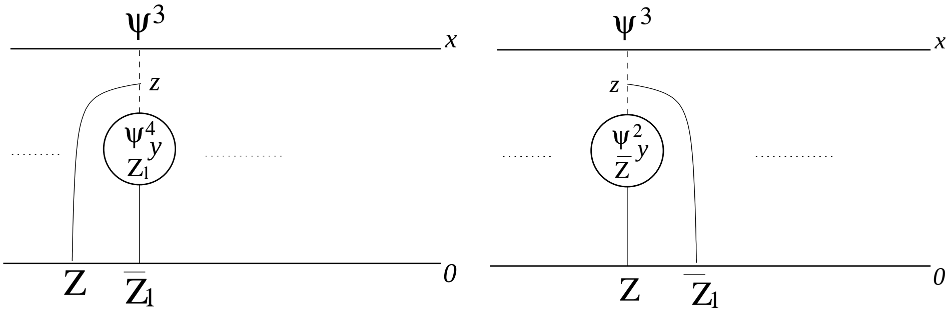

Figure 1: Diagrams contributing in the one loop calculation of (3.3). The solid lines denote scalar propagators while the dashed ones fermion propagators.

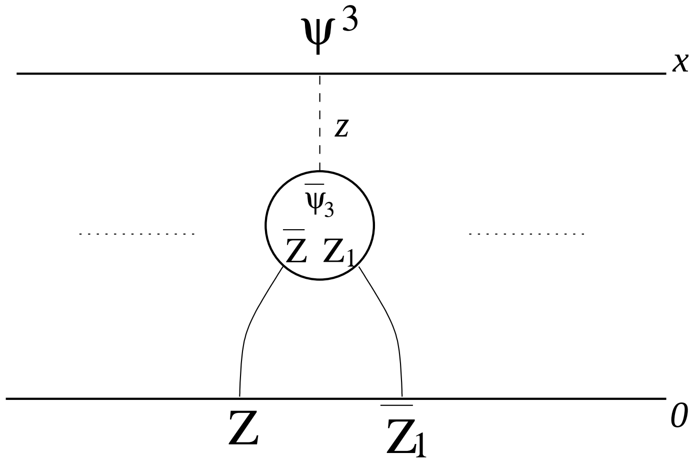

Figure 2: This is the diagram contributing to (3.6)

and demand that it is compatible with (3.2). In equation (3.3) we have written explicitly the free indices of the operators which are not scalars under the SU(N) color group. The 3-point function of (3.3) at order g receives contribution from the three terms in (A.17a). In particular, the contributions related to the last and the penultimate term of (A.17a) are depicted in figure 1 and 2 respectively. Finally, the contribution of the second term in figure (A.17a) can be obtained from the diagrams of 1 by remembering that there is an additional derivative acting on the field $Z_1(y)$. By using the propagators and the vertices summarized in the appendix A, it is straightforward to see that the two diagrams of figure 1 yield the same integral. So we have

$$G_3^{(1)\mu} = -\,4\sqrt{2}\,\Delta(y)\,\frac{N}{2^4}\,(-\mathrm{i}2\sqrt{2})\,g\int d^4z\,2\mathrm{i}^2\,\sigma^{\nu}_{\gamma\dot\beta}\partial^z_\nu\Delta(x-z)\,\epsilon^{\dot\beta\dot\gamma}\,\sigma^{\kappa}_{\alpha\dot\gamma}\partial^z_\kappa\Delta(y-z)\,\bar\sigma^{\mu\dot\alpha\alpha}\,\Delta(z)\,. \tag{3.4}$$

Some comments are in order: the factor of $4\sqrt{2}$ comes from the current, of $-\mathrm{i}2\sqrt{2}g$ is due to the insertion of the Yukawa coupling and the last factor of 2 in the integral takes into account the two diagrams in figure 1, while the overall sign comes from the fermionic Wick contractions. Finally, adopting the conventions in (A.2), the color algebra gives, in the large N limit, the factor $\frac{N}{2^4}\delta^i_l\delta^k_j$. In formula (3.4) and in what follows we drop the tensorial SU(N) structure, which is unnecessary for our computation, by defining the quantity G_3^μ via the relation $(G_3^\mu)^{i\,k}_{j\,l} = G_3^\mu \delta^i_l \delta^k_j$.

By using the integral (B.10) one obtains from (3.4):

$$G_3^{(1)\mu} = -2gN\,\Delta(y)\,\bar{\sigma}^{\mu\dot{\alpha}\alpha}\,\sigma^\nu_{\gamma\dot{\beta}}\epsilon^{\dot{\beta}\dot{\gamma}}\,\sigma^\kappa_{\alpha\dot{\gamma}}\,\frac{1}{(4\pi^2)^2}\,\frac{y_\kappa x_\nu}{x^2y^2(x-y)^2}\;. \tag{3.5}$$

The next step is to evaluate the diagram of figure 2. This gives

$$\begin{aligned} G_3^{(2)\mu} &= 4g\frac{N}{2^3}y_\tau\bar{\sigma}^{\tau\dot{\alpha}\alpha}\,\sigma^\mu_{\alpha\dot{\beta}}\,\epsilon^{\dot{\beta}\dot{\gamma}}\,[\Delta(y)]^2\,\sigma^\nu_{\gamma\dot{\gamma}}\,\partial^x_\nu\Delta(x-y) \\ &= \frac{gN}{16\pi^2}\bar{\sigma}^{\tau\dot{\alpha}\alpha}\,\partial^y_\tau\Delta(y)\,\sigma^\mu_{\alpha\dot{\beta}}\,\epsilon^{\dot{\beta}\dot{\gamma}}\,\sigma^\nu_{\gamma\dot{\gamma}}\,\partial^x_\nu\Delta(x-y)\;. \end{aligned} \tag{3.6}$$

The final ingredient we need is the contribution of the diagrams coming from the second term of (A.17a). These give an expression very similar to that of the diagrams of figure 1:

$$\begin{aligned} G_3^{(3)\mu} &= -2\sqrt{2}\,\frac{gN}{2^4}\,(-2\sqrt{2}\,\mathrm{i})\,y_\tau\bar{\sigma}^{\tau\dot{\alpha}\alpha}\,\times \\ &\qquad \int d^4z\,2\mathrm{i}^2\,\sigma^\nu_{\gamma\dot{\delta}}\partial^z_\nu\Delta(x-z)\,\epsilon^{\dot{\delta}\dot{\gamma}}\,\partial^z_\kappa\Delta(y-z)\,\sigma^\kappa_{\beta\dot{\gamma}}\Delta(z)\sigma^\rho_{\alpha\dot{\beta}}\bar{\sigma}^{\mu\dot{\beta}\beta}\,\partial^y_\rho\Delta(y)\;. \end{aligned} \tag{3.7}$$

After some algebra and by using (B.10) one gets:

$$G_3^{(3)\mu} = 2\,gN\Delta(y)\,\bar{\sigma}^{\mu\dot{\alpha}\beta}\,\frac{1}{(4\pi^2)^2}\frac{y_\kappa x_\nu}{x^2y^2(x-y)^2}\,\sigma^\nu_{\gamma\dot{\delta}}\,\epsilon^{\dot{\delta}\dot{\gamma}}\,\sigma^\kappa_{\beta\dot{\gamma}}\;. \tag{3.8}$$

By comparing (3.8) to (3.5) one can see that they precisely cancel. We are now in position to write the final expression for G_3^μ. This reads:

$$G_3^\mu = G_3^{(2)\mu} = \frac{gN}{16\pi^2}\bar{\sigma}^{\tau\dot{\alpha}\alpha}\,\partial^y_\tau\Delta(y)\,\sigma^\mu_{\alpha\dot{\beta}}\,\epsilon^{\dot{\beta}\dot{\gamma}}\,\sigma^\nu_{\gamma\dot{\gamma}}\,\partial^x_\nu\Delta(x-y)\;. \tag{3.9}$$

It is now straightforward to find the divergence of (3.9).

$$\partial^y_\mu G_3^\mu = -\frac{\mathrm{i}gN}{16\pi^2}(\delta^{(4)}(y)\epsilon^{\dot{\alpha}\dot{\gamma}}\,\sigma^\nu_{\gamma\dot{\gamma}}\,\partial^x_\nu\Delta(x) + \ldots)\,, \tag{3.10}$$

where the dots represent a term proportional to $\delta^{(4)}(x-y)$ of which we will make no use in what follows. It is now straightforward to obtain the superconformal variation of the operator $ZZ_1(0)$ with respect to $\bar{S}_1^{\dot{\alpha}}$. By comparing (3.10) to

$$\partial^y_\mu\langle S^\mu(y)\,O_1(x)\,O(0)\rangle = -\mathrm{i}\delta^4(x-y)\langle\delta_S O_1(x)\,O(0)\rangle - \mathrm{i}\,\delta^4(y)\langle O_1(x)\,\delta_S O(0)\rangle \tag{3.11}$$

one gets:

$$\bar{S}_1^{\dot{\alpha}}Z\bar{Z}_1 = \frac{-\mathrm{i}gN}{8\pi^2}\bar{\psi}_3^{\dot{\alpha}}. \tag{3.12}$$

When the scalars are in the opposite ordering $\bar{Z}_1Z$, then the action of $\bar{S}_1$ is the same but for the overall sign: $\bar{S}_1^{\dot{\alpha}}Z\bar{Z}_1 = -\bar{S}_1^{\dot{\alpha}}\bar{Z}_1Z$. In fact from the form of the currents (A.17) and of the Yukawa couplings (A.9) it is clear that the diagrams contributing to this two

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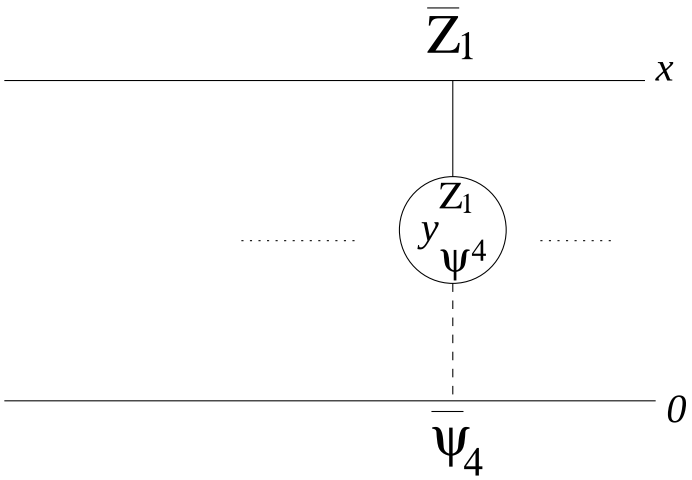

Figure 3: This diagram represent the classical variation (3.13).

cases always have a relative minus sign. Finally, when $\bar{S}_1$ acts on scalar of the same flavor, as in $\bar{S}_1^{\dot{\alpha}} Z\bar{Z}$, there is an additional factor of 1/2, which again follows from the form of the currents (A.17). So finally, by using (3.12) and these observations, one arrives at the result (3.2).

Of course, we can follow the same procedure in order to derive the classical variations. For instance, at leading order in g the action of a conformal supersymmetry on an elementary fermion is given by the diagram in figure 3 which yields

$$\bar{S}_A^{\dot{\alpha}} \bar{\psi}_{B\dot{\beta}} = 4\sqrt{2}\mathrm{i}\Phi_{AB}\delta^{\dot{\alpha}}_{\dot{\beta}}\,, \tag{3.13}$$

where again all fields are at $x = 0$

An independent check on the coefficient of equation (3.2) can be performed via the SU(2, 2|4) algebra by using only the well known expression for the spin chain Hamiltonian at order g^2 [23]. If we act with the algebra on scalars of different flavour, at order g^2 we have $\left\{\bar{Q}^{A\dot{\alpha}}, \bar{S}_B^{\dot{\beta}}\right\} = \epsilon^{\dot{\alpha}\dot{\beta}}\,\delta^A_B\,2\mathbb{H}$, with $\mathbb{H} = \frac{g^2N}{8\pi^2}\,(\mathbb{I} - \mathbb{P})$. In particular, we can restrict ourselves to

$$\left\{\bar{Q}^{1\dot{\alpha}}, \bar{S}_1^{\dot{\beta}}\right\} Z_2 Z_3 \;=\; \epsilon^{\dot{\alpha}\dot{\beta}}\,\frac{g^2N}{4\pi^2}\,[Z_2, Z_3]\,. \tag{3.14}$$

On the other hand, since the action of $\bar{Q}^1$ on Z_2Z_3 is zero both at classical level and at order g^2, where it is forbidden by the SU(4) symmetry, the left hand side of the previous equation reduces to the action of $\bar{Q}^{1\dot{\alpha}}\bar{S}_1^{\dot{\beta}}$ on the pair of fields, which can be computed via (3.2), giving exactly the right hand side of (3.14).

We can now use (3.2) and (3.13) to compute the first quantum correction to the highest weight state (3.1). Again for the sake of concreteness, let us focus on the superconformal charge $\bar{S}_1$. By using (3.2) on the operator (3.1), one can see that the variations involving

$\bar{Z}_1$ yields

$$\frac{\mathrm{i}gN}{8\pi^2}\sqrt{\frac{8\pi^2}{N}}\sqrt{\frac{N_0^{-J-1}}{(J+3)}}\sum_{p=0}^{J-1}\left(\cos\frac{\pi n(2p+3)}{J+3}-\cos\frac{\pi n(2p+5)}{J+3}\right)\mathrm{Tr}\big[Z_1Z^p\bar{\psi}_3^{\dot{\alpha}}Z^{J-1-p}\big]\,, \tag{3.15}$$

where we have explicitly written one of the N_0 factors, because the resulting operators has only $J+1$ fields, one less in comparison to the original operator (3.1). Eq. (3.15) can be rewritten as follows

$$\mathrm{i}\,\frac{g\sqrt{N}}{\sqrt{2}\pi}\sqrt{\frac{N_0^{-J-1}}{(J+3)}}\sin\frac{\pi n}{J+3}\sum_{p=0}^{J-1}\sin\frac{\pi n(2p+4)}{J+3}\mathrm{Tr}\big[Z_1Z^p\bar{\psi}_3^{\dot{\alpha}}Z^{J-1-p}\big]\,. \tag{3.16}$$

From (3.2) it is clear that the action of any (1-loop corrected) superconformal charge on two identical scalar is trivial. The action of $\bar{S}_1$ on the couple ZZ_2 yields similar terms involving $\bar{\psi}_4$, instead of $\bar{\psi}_3$.

Exactly as it happened in the string theory computation, we can cancel these order g contributions against the classical variation of a term containing two fermionic impurities, but which is suppressed by an explicit factor of g. The result in (3.16) suggests to consider the following form for the highest weight state

$$\begin{aligned}\mathcal{O}_n^J \;=\; & \sqrt{\frac{N_0^{-J-2}}{(J+3)}}\sum_{i=1}^{3}\sum_{p=0}^{J}\cos\frac{\pi n(2p+3)}{J+3}\mathrm{Tr}\big[Z_iZ^p\bar{Z}_iZ^{J-p}\big] \\ & +\frac{g\sqrt{N}}{4\pi}\sin\frac{\pi n}{J+3}\sqrt{\frac{N_0^{-J-1}}{(J+3)}}\sum_{p=0}^{J-1}\sin\frac{\pi n(2p+4)}{J+3}\mathrm{Tr}\big[\psi^{1\alpha}Z^p\psi_\alpha^2Z^{J-1-p}\big] \\ & -\frac{g\sqrt{N}}{4\pi}\sin\frac{\pi n}{J+3}\sqrt{\frac{N_0^{-J-1}}{(J+3)}}\sum_{p=0}^{J-1}\sin\frac{\pi n(2p+4)}{J+3}\mathrm{Tr}\big[\bar{\psi}_{4\dot{\alpha}}Z^p\bar{\psi}_3^{\dot{\alpha}}Z^{J-1-p}\big]\,.\end{aligned} \tag{3.17}$$

The coefficients in the second and third line have been fixed in order to satisfy (1.1). In fact, we have seen that there is a quantum contribution from the first line summarized in (3.2), and there are classical contributions from the new terms summarized in (3.13). If we keep focusing on $\bar{S}_1$, (3.16) summarizes the quantum contribution from the planar action on the Z and Z_1 which is canceled by the classical variation of $\bar{\psi}_4$ in the last line. Similar computations show that this pattern applies also to the action of the other conformal supercharges and to the action on the other (couple of) fields. Notice also that the contribution of the boundary terms of (3.1) (i.e. those with $p=0$ and $p=J$) due to the action of $\bar{S}_1$ on $Z_1\bar{Z}_1$ and $\bar{Z}_1Z_1$ sum to zero. In fact, the diagrams involving respectively the pairs $Z_i\bar{Z}_i$ and $\bar{Z}_iZ_i$ come with the same phase factor but opposite sign due to (3.2).

We close this section with some comments regarding the form of the primary operator we have derived. Clearly the result (3.17) requires to have $J\geq 1$; if $J=0$, then the highest weight state is the standard Konishi operator and no mixing is present since it is not possible to build other SU(4) scalars with the same free scaling dimension. A second observation is that in the two-point function $\langle\bar{\mathcal{O}}_n^J\mathcal{O}_n^J\rangle$ there is no overlap at one loop between the leading and the subleading term of (3.17). In fact the first possible diagram involves three Yukawa couplings and therefore it is of order g^3. Thus, eq. (3.17) can be used as a non trivial test for H_3, the cubic term of the Hamiltonian for the full $P\,\mathrm{SU}(2,2|4)$ theory, which has not been computed yet. H_3 should capture the mixing in (3.17) for $p = 0$. In order to capture the terms with $p \geq 1$ one would need to compute the two-point function (or the Hamiltonian) at higher orders in g. Moreover, in the large J limit the corrections to the naive form of the primary, although present, do not alter the anomalous dimension calculated in [24]. This can be easily seen since any contribution to the 2-point correlator of the primary coming from terms like $\langle \mathrm{Tr}\big[\phi^{\mathrm{i}} Z^l \phi^{\mathrm{i}} Z^{J-l}\big] \mathrm{Tr}\big[\psi Z^l \psi Z^{J-1-l}\big]\rangle$ should involve both impurities and is thus suppressed in the large J limit. Also in the computation of the 1-loop anomalous dimension, all the corrections terms in (3.2) can be neglected even at finite J. Finally from (3.17) it is possible to derive the descendant operators , either by the same method employed here for the primary, with the only difference that one has to use the supersymmetry current instead of the superconformal one, or by using the Konishi anomaly [7].

4. Comparison with the predictions from string theory

In section 3 we saw that the mixing between scalar and fermion impurities in the $\mathcal{N} = 4$ SYM primary we considered mirrors exactly the patterns we found in the correspondent string theory computation. Actually, if we expand the string theory highest weight state in (2.11) in powers of $(\mu\alpha)^{-1}$, the first subleading term, quadratic in the fermionic oscillators, matches exactly with the large J expansion of the second and third line in (3.17). In particular, up to order $\frac{1}{\mu\alpha} = \sqrt{\lambda'}$, the string state (2.11) becomes:

$$|n\rangle \approx \frac{1}{4}\left[a^{\dagger}{}^{i'}_{n} a^{\dagger}{}^{i'}_{n} + a^{\dagger}{}^{i'}_{-n} a^{\dagger}{}^{i'}_{-n} + n\sqrt{\lambda'}\left(b^{\dagger}{}_{-n}\mathbb{P}^{+} b^{\dagger}{}_{n} - b^{\dagger}{}_{-n}\mathbb{P}^{-} b^{\dagger}{}_{n}\right)\right]|\alpha\rangle \,. \tag{4.1}$$

We can translate the bosonic and fermionic contributions of this formula into the corresponding BMN operators. In this way, we can read from (4.1) a prediction for the BMN limit of the first and second terms of the quantum corrected SYM highest weight operator. Let us start from the term in (4.1) with the bosonic oscillator. According to the standard PP-wave/BMN dictionary we have

$$\frac{1}{4}\left[a^{\dagger}{}^{i'}_{n} a^{\dagger}{}^{i'}_{n} + a^{\dagger}{}^{i'}_{-n} a^{\dagger}{}^{i'}_{-n}\right]|\alpha\rangle \;\leftrightarrow\; \mathcal{O}_n^{(0)J}\,, \tag{4.2}$$

where $\mathcal{O}_n^{(0)J}$ is defined in (3.1). The string state on the l.h.s. is normalized to one as it is the tree-level 2-point function of the corresponding gauge theory operator

$$\langle \bar{\mathcal{O}}_n^{(0)J}(x)\mathcal{O}_n^{(0)J}(y)\rangle = \frac{(-1)^{J+2}}{(x-y)^{2(J+2)}}, \tag{4.3}$$

The term with the fermionic oscillators in (4.1) correspond to a gauge theory operator with two spinors:

$$\frac{1}{2}\left[b^{\dagger}{}_{-n}\mathbb{P}^{+} b^{\dagger}{}_{n}\right]|\alpha\rangle \;\leftrightarrow\; \mathcal{O}_n^{(1)J} = \frac{1}{2}\sqrt{\frac{N_0^{-J-1}}{J+1}}\sum_{p=0}^{J-1}\sin\frac{\pi n(2p+2)}{J+1}\mathrm{Tr}\big[\psi^{1\alpha} Z^p \psi^2_{\alpha} Z^{J-p-1}\big] \tag{4.4}$$

and the similar formula relates the $\mathbb{P}^-$ projection of the string state to a gauge theory operator with the spinors $\bar\psi_3$ and $\bar\psi_4$. Also in this case the normalizations have been fixed by requiring that the SYM tree-level 2-point function takes the canonical form (4.3) and that the norm of the string state is one.

Then rewriting $\sqrt{\lambda'} = \frac{\sqrt{N} g_{YM}}{J}$ we get that the gauge theory operator corresponding to the string state (4.1) is:

$$
\begin{aligned}
(\mathcal{O}_{st})_n^J \, = & \sqrt{\frac{N_0^{-J-2}}{J+3}} \sum_{p=0}^{J} \cos\frac{\pi n(2p+3)}{J+3} \mathrm{Tr}\left[Z_i Z^p \bar Z_i Z^{J-p}\right] + \\
& + \frac{g\sqrt{N} n}{4J} \sqrt{\frac{N_0^{-J-1}}{J+1}} \sum_{p=0}^{J-1} \sin\frac{\pi n(2p+2)}{J+1} \mathrm{Tr}\left[\psi^{1\alpha} Z^p \psi^2_\alpha Z^{J-p-1}\right] + \\
& - \frac{g\sqrt{N} n}{4J} \sqrt{\frac{N_0^{-J-1}}{J+1}} \sum_{p=0}^{J-1} \sin\frac{\pi n(2p+2)}{J+1} \mathrm{Tr}\left[\bar\psi_{3\dot\alpha} Z^p \bar\psi_4^{\dot\alpha} Z^{J-p-1}\right] .
\end{aligned} \tag{4.5}
$$

In the large J limit this result agrees with (3.17) derived in section (3) by using perturbative field theory.

5. Conclusions

In this paper, we derived the expression of the two impurities highest weight state for type IIB string theory on the maximal supersymmetric PP-wave background. Our string result is valid at tree-level, but is exact in $\mu\alpha$. Moreover, we used the superconformal properties of $\mathcal{N}=4$ SYM to derive the two-impurity highest weight state of the $\mathrm{SU}(2,2|4)$ superconformal algebra at finite J and up to the order g. In both cases, the naive form of the state is corrected by a term containing two fermionic impurities. Then we showed that the large-J limit of the gauge theory highest weight state matches with the strong curvature expansion of the string state to the same order in perturbation theory. This precise agreement is partially unexpected, as the BMN string theory is valid in the limit $\lambda, J \to \infty$ with λ' fixed, while perturbative gauge theory computations require $\lambda \ll 1$. We do not know any clear reason why these two different scaling limits should match exactly. However this happens for the planar anomalous dimensions up to order g^6 [25, 26]. From our results it follows that also the form of the operators, not just their dilatation eigenvalue, matches up to order g.

It would be certainly interesting to check whether the agreement between the string and the field theory highest weight states survives at the next order $\mathcal{O}(g^2)$. In order to check this, one would need to know more about the quantum corrected form of the conformal supercharges. This can done by following the approach discussed in section 3 and by deriving the quantum corrected action of the S and $\bar S$ on fermion and vector fields. For instance, there will be certainly a contribution of the form $\bar S \psi \Phi \sim \partial\!\!\!/ \Phi$. This will induce on the field theory side the same pattern we have seen in section 2 and thus we expect that, at the order g^2, also derivative impurities appear in the explicit form of the field theory highest weight state. On the other hand our field theory result is exact in

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J. It would be interesting to consider the subleading corrections in the large-J limit. These should be captured by the near-BMN limit of the $AdS_5 \times S^5$ string theory which has been thoroughly studied [32]. In particular, the form of the string supercharges in the near-BMN limit is known in the literature [33]. Taking into account these corrections on the string side would allow another comparison with our gauge theory result. Another possible extension of the approach proposed here is to consider the non-planar action of the conformal supercharges. It would be interesting to see whether this is sufficient to reproduce the $1/N$ mixing obtained in [4, 5] by using the standard approach of the orthonormalization of the 2-point correlators.

As pointed out in section 3, the corrections we derived to the field theory operators are not needed in the computation of the $\mathcal{O}(g^2)$ anomalous dimensions and also can be neglected in the BMN limit at all orders. However, we expect that they play a crucial role in the computation of three and higher point correlators. It would be certainly interesting to see this in some explicit examples. This would provide a systematic basis to study the correspondence between BMN string theory and gauge theory in presence of fermionic impurities and generalize the results of [27, 28].

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A. $\mathcal{N} = 4$ SYM conventions

The Lagrangian and the supersymmetry variations of the four dimensional $\mathcal{N} = 4$ SYM can be derived by dimensional reduction from the ten dimensional $N = 1$ SYM theory [29]. Here we recall the main steps of this derivation mainly with the aim of setting up some notations that are useful for building the dictionary between BMN string states and gauge theory operators.

The ten-dimensional action is

$$S_{10} = \int d^{10}x Tr \left[-\frac{1}{2} F_{MN} F^{MN} + \mathrm{i}\bar{\lambda}\Gamma^M D_M \lambda \right] \tag{A.1}$$

We adopt the "mostly-minus" metric $(+, -, \ldots, -)$ and the following conventions for the gauge group generators:

$$\mathrm{Tr}\big[T^a T^b\big] = \frac{\delta^{ab}}{2}, \quad \Big[T^a, T^b\Big] = i f^{abc} T^c, \quad (T^a)^i_j (T^a)^k_l = \frac{1}{2}\left(\delta^i_l \delta^k_j - \frac{1}{N}\delta^i_j \delta^k_l\right) \tag{A.2}$$

A useful representation of the the ten dimensional Gamma matrices with mostly minus signature is

$$\Gamma^\mu = 1_8 \otimes \gamma^\mu \ , \quad \Gamma^{i+3} = \sigma^1 \otimes \eta^i \otimes \gamma^5 \ , \quad \Gamma^{i+6} = -\sigma^2 \otimes \bar{\eta}^i \otimes \gamma^5 \,, \tag{A.3}$$

where 1_n is the $n \times n$ identity matrix, the γ^μ's are the standard four dimensional gamma matrices in the Weyl representation

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \quad \text{and} \quad \gamma^5 = \mathrm{i} \prod_{j=0}^{3} \gamma^j = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{A.4}$$

with $\sigma^0 = \bar{\sigma}^0 = 1_2$ is, while $\sigma^i = -\bar{\sigma}^i$ are the standard Pauli matrices.

The σ-matrices satisfy the following relation:

$$\sigma^\mu_{\alpha\dot{\alpha}}\, \sigma^\nu_{\beta\dot{\beta}}\, \epsilon^{\dot{\alpha}\dot{\beta}} \;=\; -\eta^{\mu\nu}\epsilon_{\alpha\beta} \;+\; 2\sigma^{\mu\nu}_{\alpha\beta} \tag{A.5}$$

where we have defined $\sigma^{\mu\nu}_{\alpha\beta} = \frac{1}{4}\left(\sigma^\mu_{\alpha\dot{\alpha}}\, \sigma^\nu_{\beta\dot{\beta}} \;-\; \sigma^\nu_{\alpha\dot{\alpha}}\, \sigma^\mu_{\beta\dot{\beta}}\right)\epsilon^{\dot{\alpha}\dot{\beta}}$, and $\epsilon_{12} = \epsilon^{21} = 1$.

Finally η^i , $\bar{\eta}^i$ are the 't Hooft matrices

$$\eta^i_{AB} \;=\; \delta_{iA}\delta_{B4} - \delta_{iB}\delta_{A4} + \epsilon_{iAB4}, \tag{A.6a}$$

$$\bar{\eta}^i_{AB} \;=\; \delta_{iA}\delta_{B4} - \delta_{iB}\delta_{A4} - \epsilon_{iAB4}, \tag{A.6b}$$

which satisfy $\eta^i\eta^j = -\delta^{ij}1_4 - \epsilon^{ijk}\eta^k$ and $\bar{\eta}^i\bar{\eta}^j = -\delta^{ij}1_4 + \epsilon^{ijk}\bar{\eta}^k$.

In this representation we have $\Gamma^{11} = \sigma^3 \otimes 1_4 \otimes \gamma^5$. The gaugino of the ten dimensional theory λ is a Majorana-Weyl spinor ($\Gamma^{11}\lambda = \lambda$) which, with the conventions above can be express in term of the four Weyl plus four anti-Weyl spinors of the four dimensional theory, ψ^A_α and $\bar{\psi}^{\dot{\alpha}}_A$ respectively:

$$\lambda^t = \left[\left(0, \bar{\psi}^{\dot{\alpha}}_{A=1}\right), \ldots, \left(0, \bar{\psi}^{\dot{\alpha}}_{A=4}\right),\ \left(\psi^{A=1}_\alpha, 0\right), \ldots, \left(\psi^{A=1}_\alpha, 0\right)\right], \tag{A.7}$$

The action of the chirality matrix gives $\gamma^5\,{}^t(0, \bar{\psi}^{\dot{\alpha}}) = {}^t(0, \bar{\psi}^{\dot{\alpha}})$ and $\gamma^5\,{}^t(\psi_\alpha, 0) = -{}^t(\psi_\alpha, 0)$.

The index A rotates into representations of the internal SU(4) R-symmetry of the four dimensional $\mathcal{N} = 4$ SYM theory. In particular, the Weyl spinor ψ^A_α transform in the fundamental representation, while their conjugates $\bar{\psi}^{\dot{\alpha}}_A$ transform in the antifundamental one.

The six scalar fields arising from the internal components of the gauge field can be organized into the components Φ_{AB} of a tensor in the antisymmetric representation of SU(4):

$$\Phi_{AB} = \frac{1}{2\sqrt{2}} \sum_{j=1}^{3} \left[A_{j+3}\eta^j_{AB} + \mathrm{i}A_{j+6}\bar{\eta}^j_{AB}\right]. \tag{A.8}$$

With these conventions, the dimensionally reduced Lagrangian is

$$\begin{aligned} L = Tr\Big[&-\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + 2D_\mu\Phi_{AB}D^\mu\Phi^{AB} + 2i\psi^{\alpha A}\sigma^\mu_{\alpha\dot{\alpha}}(D_\mu\bar{\psi}^{\dot{\alpha}}_A) + \\ &2g^2\left[\Phi^{AB}, \Phi^{CD}\right]\left[\Phi_{AB}, \Phi_{CD}\right] - g2\sqrt{2}\left(\left[\psi^{\alpha A}, \Phi_{AB}\right]\psi^B_\alpha - \left[\bar{\psi}_{\dot{\alpha}A}, \Phi^{AB}\right]\bar{\psi}^{\dot{\alpha}}_B\right)\Big] \end{aligned} \tag{A.9}$$

where the scalar fields with upper indices Φ^{AB} are defined as follow:

$$\Phi^{AB} = \frac{1}{2}\epsilon^{ABCD}\Phi_{CD} \equiv \Phi^*_{AB}, \tag{A.10}$$

and the covariant derivative is $D_\mu\phi = \partial_\mu\phi - ig[A_\mu,\phi]$.

Out of (A.9) we read the Minkowskian free scalar propagator:

$$\langle Z_i^a(x)\,\bar Z_j^b(y)\rangle \;=\; \delta_{ij}\delta^{ab}\Delta_{xy} \qquad \Box_x\Delta_{xy} = -\mathrm{i}\delta^4(x-y) \tag{A.11}$$

and the free fermionic one:

$$\langle\psi_\alpha^{Aa}(x)\bar\psi_{\dot\alpha B}^{b}(y)\rangle \;=\; \mathrm{i}\delta^{ab}\delta_B^A\sigma^\mu_{\alpha\dot\alpha}\partial^x_\mu\Delta_{xy} \quad\Rightarrow\quad \langle\bar\psi_A^{\dot\alpha a}(x)\psi^{\alpha Bb}(y)\rangle \;=\; \mathrm{i}\delta^{ab}\delta_B^A\bar\sigma^{\mu\,\dot\alpha\alpha}\partial^x_\mu\Delta_{xy}, \tag{A.12}$$

where $\Delta_{xy} = -\frac{1}{4\pi^2\,(x-y)^2}$, and the σ-matrices are defined after formula (A.4).

The 10D supersymmetry transformation, $\delta A_M = \mathrm{i}\bar\xi\Gamma_M\lambda$ and $\delta\lambda = \frac14[\Gamma^M,\Gamma^N]F_{MN}\xi$, decompose as follows:

$$\begin{aligned}
\bar\xi \;&=\; \big[\big(\xi^{\alpha\,A=1},0\big),\dots,\big(\xi^{\alpha\,A=4},0\big),\big(0,\bar\xi_{\dot\alpha\,A=1}\big),\dots,\big(0,\bar\xi_{\dot\alpha\,A=4}\big)\big]\\
\delta\Phi_{AB} \;&=\frac{\mathrm{i}}{\sqrt2}\left[\epsilon_{ABCD}\,\xi^{C\,\alpha}\,\psi^D_\alpha \;-\; \bar\xi_{A\,\dot\alpha}\bar\psi_B^{\dot\alpha} \;+\; \bar\xi_{B\,\dot\alpha}\bar\psi_A^{\dot\alpha}\right] && \text{(A.13a)}\\
\delta\psi^A_\alpha \;&=\sigma^{\mu\nu}{}_\alpha{}^\beta\,\xi^A_\beta\,F^{\mu\nu} \;+\; 2\sqrt2\sigma^\mu{}_{\alpha\dot\alpha}\,\bar\xi_B^{\dot\alpha}\,D_\mu\Phi^{AB} \;-\; 4\mathrm{i}g\left[\Phi^{AC},\Phi_{CB}\right]\,\xi^B_\alpha && \text{(A.13b)}\\
\delta\bar\psi_A^{\dot\alpha} \;&=\bar\sigma^{\mu\nu\,\dot\alpha}{}_{\dot\beta}\,\bar\xi^{\dot\beta}_A\,F^{\mu\nu} \;-\; 2\sqrt2\bar\sigma^{\mu\,\dot\alpha\alpha}\,\xi^B_\alpha\,D_\mu\Phi_{AB} \;-\; 4\mathrm{i}g\left[\Phi_{AC},\Phi^{CB}\right]\,\bar\xi^{\dot\alpha}_B && \text{(A.13c)}\\
\delta A_\mu \;&=\mathrm{i}\left(\sigma_{\mu\,\alpha\dot\alpha}\xi^{\alpha A}\bar\psi_A^{\dot\alpha} \;+\; \bar\sigma_\mu^{\dot\alpha\alpha}\bar\xi_{\dot\alpha A}\psi^A_\alpha\right) && \text{(A.13d)}
\end{aligned}$$

The supercurrent associated to the invariance of the ten-dimensional theory under supersymmetry transformations is:

$$Q^M = \frac{\mathrm{i}}{2}\left[\Gamma^R,\Gamma^N\right]\mathrm{Tr}\big[F_{RN}\Gamma^M\lambda\big], \quad M=0,\dots,9 \tag{A.14}$$

After dimensional reduction to $D=4$, we get the following supersymmetric current for $\mathcal N = 4$ SYM:

$$Q^\mu = {}^t\left[\left(Q^\mu_{\alpha A=1},0\right),\dots\left(Q^\mu_{\alpha A=4},\,0\right),\,\left(0,\,\bar Q^{\mu\,\dot\alpha A=1}\right),\dots\left(0,\,\bar Q^{\mu\,\dot\alpha A=4}\right)\right] \tag{A.15}$$

with

$$Q^\mu{}_{\alpha A} = 2\mathrm{i}\mathrm{Tr}\big[(\sigma^{\rho\nu})_\alpha^\beta F_{\rho\nu}\sigma^\mu_{\beta\dot\beta}\bar\psi_A^{\dot\beta} + 2\sqrt2 D_\rho\Phi_{AB}\sigma^\rho_{\alpha\dot\alpha}\bar\sigma^{\mu\,\dot\alpha\beta}\psi^B_\beta - 4\mathrm{i}g[\Phi_{AC},\Phi^{CB}]\sigma^\mu_{\alpha\dot\alpha}\bar\psi_B^{\dot\alpha}\big] \tag{A.16a}$$

$$\bar Q^{\mu\,\dot\alpha A} = 2\mathrm{i}\mathrm{Tr}\big[(\bar\sigma^{\rho\nu})^{\dot\alpha}_{\dot\beta}F_{\rho\nu}\bar\sigma^{\mu\,\dot\beta\beta}\psi^A_\beta - 2\sqrt2 D_\rho\Phi^{AB}\bar\sigma^{\rho\,\dot\alpha\alpha}\sigma^\mu_{\alpha\dot\beta}\bar\psi_B^{\dot\beta} - 4\mathrm{i}g[\Phi^{AC},\Phi_{CB}]\bar\sigma^{\mu\dot\alpha\alpha}\psi^B_\alpha\big] \tag{A.16b}$$

On the other hand, the current associated to the superconformal transformations are obtained first by replacing, in the supersymmetry variation of a field, the supersymmetry parameters $\xi^{\alpha A}$ and $\bar\xi_{\dot\alpha A}$ with $\mathrm{i}x_\mu\bar\sigma^{\mu\,\dot\alpha\alpha}\bar\zeta^A_{\dot\alpha}$ and $\mathrm{i}x_\mu\sigma^\mu_{\alpha\dot\alpha}\zeta^\alpha_A$ respectively, and then adding all possible x-independent terms with the same mass dimension and the same quantum numbers which are compatible with the superconformal algebra [30]. Out of this process, we get the following superconformal currents:

$$\begin{aligned}
\bar S^{\mu\dot\alpha}_{\;\;A} = 2x_\tau(\bar\sigma^\tau)^{\dot\alpha\alpha}\mathrm{Tr}\big[&(\sigma^{\rho\nu})_\alpha^\beta F_{\rho\nu}\sigma^\mu_{\beta\dot\beta}\bar\psi_A^{\dot\beta} + 2\sqrt2 D_\rho\Phi_{AB}\sigma^\rho_{\alpha\dot\beta}\bar\sigma^{\mu\,\dot\beta\beta}\psi^B_\beta +\\
&- 4\mathrm{i}g\left[\Phi_{AC},\Phi^{CB}\right]\sigma^\mu_{\alpha\dot\beta}\bar\psi_B^{\dot\beta}\big] + 8\sqrt2\mathrm{Tr}\big[\phi_{AB}(\bar\sigma^\mu)^{\dot\alpha\alpha}\psi^B_\alpha\big],
\end{aligned} \tag{A.17a}$$

$$S^{\mu A}_{\ \alpha} = 2x_\tau \sigma^\tau_{\alpha\dot\alpha}\mathrm{Tr}\big[(\bar\sigma^{\rho\nu})^{\dot\alpha}_{\ \dot\beta}F_{\rho\nu}\bar\sigma^{\mu\,\dot\beta\beta}\psi^A_\beta - 2\sqrt{2}D_\rho\Phi^{AB}\bar\sigma^{\rho\,\dot\alpha\beta}\sigma^\mu_{\beta\dot\beta}\bar\psi^{\dot\beta}_B + \\ - 4\mathrm{i}g[\Phi^{AC},\Phi_{CB}]\bar\sigma^{\mu\dot\alpha\beta}\psi^B_\beta\big] - 8\sqrt{2}\mathrm{Tr}\big[\phi^{AB}\sigma^\mu_{\alpha\dot\alpha}\bar\psi^{\dot\alpha}_B\big] \quad \text{(A.17b)}$$

where the coefficients of the last terms have been fixed by requiring that $\partial_\mu S^\mu = 0$ and $\partial_\mu \bar S^\mu = 0$ on-shell.

We refer to section 3 of [22] for the four-dimensional superconformal algebra closed by the charges associated to the currents in (A.15) and (A.17), together with the generator of the conformal algebra in four dimensions.

At this point it is straightforward also to make contact with the $\mathcal{N} = 1$ formalism where the scalars are arranged into three complex fields

$$Z_i = (A_{i+3} + \mathrm{i}A_{i+6})/\sqrt{2} \quad \text{(A.18)}$$

$$\Phi_{14} = \frac{1}{2}Z_1, \quad \Phi_{24} = \frac{1}{2}Z_2, \quad \Phi_{34} = \frac{1}{2}Z_3, \quad \Phi_{13} = -\frac{1}{2}\overline{Z}_2\,, \quad \Phi_{23} = \frac{1}{2}\overline{Z}_1, \quad \Phi_{12} = \frac{1}{2}\overline{Z}_3.$$

We select the $\mathrm{U}(1) \in \mathrm{SU}(4)$ which rotates $Z_3 \doteq Z$ as the BMN $\mathrm{U}(1)$. In order to see the fate of the various spinors in the BMN limit, it is convenient to compute their charges under the Cartan generators $(J^{(s)}_{Z_1}, J^{(s)}_{Z_2}, J^{(s)}_{Z_3})$, with $J^{(s)}_{Z_i} = -\mathrm{i}\Gamma^{i+3}\Gamma^{i+6}$, that rotate the complex scalars (Z_1, Z_2, Z_3)

$$\psi^1 \to \left(-\frac{1}{2},\frac{1}{2},\frac{1}{2}\right), \ \psi^2 \to \left(\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right), \ \psi^3 \to \left(\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right), \ \psi^4 \to \left(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right). \quad \text{(A.19)}$$

Of course $\bar\psi_A$ has the opposite assignments. The supersymmetries $Q_{\alpha\,A=1,2}$ have the same chirality both in in the space-time and in the Z_1, Z_2 directions and they correspond in the PP-wave string theory to the kinematical generators $\mathbb{P}^+Q^+/2$. Similarly the supersymmetries $\bar Q^{\dot\alpha\,A=3,4}$ correspond in to the other kinematical generators $\mathbb{P}^-\bar Q^+/2$. This assignment is also consistent with the fact that these supersymmetries act non-trivially on Z_3 and thus generate the BPS multiplet starting from the operator $\mathrm{Tr}\big[Z^J\big]$ in field theory or the vacuum state $|p^+\rangle$ in the string theory language. The other supersymmetry variations have opposite chirality in the space-time and in the internal Z_1, Z_2 plane and thus must correspond to the dynamical supercharges Q^- and $\bar Q^-$.

B. A useful integral

The computation of the one-loop correlators in section 3 involve the following integration (let us remember that $\sigma^\mu = (1_2, \sigma^i_{(P)}), i = 1, 2, 3$):

$$\begin{aligned} I^M(x_1; x_2, x_3) \ &\doteq \sigma^\mu_{\alpha\dot\alpha}\sigma^\nu_{\beta\dot\beta}\epsilon^{\dot\alpha\dot\beta}\int d^4x \frac{1}{(x_1-x)^2}\partial^{x_2}_\mu\frac{1}{(x_2-x)^2}\partial^{x_3}_\nu\frac{1}{(x_3-x)^2} \\ &= -\,\mathrm{i}4\pi^2\sigma^\mu_{\alpha\dot\alpha}\sigma^\nu_{\beta\dot\beta}\epsilon^{\dot\alpha\dot\beta}\frac{x_{12\mu}x_{13\nu}}{x^2_{12}x^2_{13}x^2_{23}} \end{aligned} \quad \text{(B.1)}$$

The last identity follows by first analytically continuing the integral to Euclidean space-time, and then computing separately the symmetric and antisymmetric components in the spacetime indices. The computation of the symmetric part is straightforward, while the antisymmetric one is evaluated by connecting it to another conformal integral, this time in $d=6$ (Euclidean) dimensions [31]:

$$I_6(y_1,y_2,y_3) \doteq \int d^6y \left[\frac{1}{(y_1-y)^2\,(y_2-y)^2\,(y_3-y)^2}\right] = \pi^3 \frac{1}{y_{12}^2\,y_{13}^2\,y_{23}^2}. \tag{B.2}$$

Introducing $x_M^0 \doteq -\mathrm{i}x_E^4$, we can rewrite (B.1) as

$$\begin{aligned} I^M(x_1;x_2,x_3) =& -\mathrm{i}\sigma^m_{(E)\,\alpha\dot\alpha}\sigma^n_{(E)\,\beta\dot\beta}\epsilon^{\dot\alpha\dot\beta}\int_E d^4x \frac{1}{(x_1-x)^2}\partial_m^{x_2}\frac{1}{(x_2-x)^2}\partial_n^{x_3}\frac{1}{(x_3-x)^2} \\ \doteq& -\mathrm{i}\sigma^m_{(E)\,\alpha\dot\alpha}\sigma^n_{(E)\,\beta\dot\beta}\epsilon^{\dot\alpha\dot\beta} I^E_{mn}(x_1;x_2,x_3) \end{aligned} \tag{B.3}$$

where we have defined $\sigma^m_{(E)} = (\mathrm{i}\sigma^j_{(P)}, -1_2), j=1,2,3.$

Remembering that $\sigma^m_{(E)\,\alpha\dot\alpha}\sigma^n_{(E)\,\beta\dot\beta}\epsilon^{\dot\alpha\dot\beta} = -\delta^{mn}\epsilon_{\alpha\beta}+2\sigma^{mn}_{\alpha\beta}$, we can decompose the integral into its symmetric plus antisymmetric part. The symmetric part has been computed in [28], the result being $\delta^{mn}I^E_{mn}(x_1;x_2,x_3) = 4\pi^2\frac{x_{12\,m}x_{13\,n}\delta^{mn}}{x_{12}^2x_{13}^2x_{23}^2}$. The antisymmetric one can be evaluated via a comparison with the integral in equation (B.2), after writing both of them in term of Schwinger parameters. We get:

$$I_6 = \int d^6y \int_0^\infty \left(\prod_{i=1}^3 \frac{d\alpha_i\,\alpha_i}{\Gamma(2)}\right) e^{-\sum_i \alpha_i (y_i-y)^2}, \tag{B.4}$$

which, after the Gaussian integration over y and after introducing the new integration variables

$$\hat\alpha_i = \frac{\alpha_i}{T}, \quad i=1,2,3, \quad \text{with } \; T=\sum_i \alpha_i \tag{B.5}$$

can be rewritten as:

$$I = \pi^3 \int_0^\infty dT\,T^2 \int_0^1 \left(\prod_i d\hat\alpha_i\,\hat\alpha_i\right)\delta^d\Big(1-\sum_i\hat\alpha_i\Big)e^{-T\sum_{i<j}\hat\alpha_i\hat\alpha_j\,y_{ij}^2}. \tag{B.6}$$

On the other hand, in term of Schwinger parameters $I^E_{[mn]}(x_1;x_2,x_3)$ becomes:

$$I^E_{[mn]}(x_1;x_2,x_3) = \int d^4u\,\partial^{x_2}_{[m}\partial^{x_3}_{n]}\int_0^\infty d\alpha_i e^{-\sum_i \alpha_i(x_i-x)^2}. \tag{B.7}$$

The integrand can be simplified by taking the derivatives and exploiting the antisymmetry in the spacetime indices. Then, performing the change of variables in (B.5) and the Gaussian integration over x^m we get:

$$I^E_{[mn]}(x_1;x_2,x_3) = 4\pi^2 x_{12[m}x_{13n]}\int_0^\infty dT\,T^2\int_0^1\left(\prod_{i=1}^3 d\hat\alpha_i\,\alpha_i\right)\delta^d\Big(1-\sum_i\hat\alpha_i\Big)e^{-T\sum_{i<j}\hat\alpha_i\hat\alpha_j x_{ij}^2}. \tag{B.8}$$

The direct comparison between (B.8) and (B.6) shows immediately:

$$I^E_{[mn]}(x_1;x_2,x_3) = \frac{1}{\pi}I_6(y_1,y_2,y_3)\big|_{y_{ij}^2\doteq x_{ij}^2} = 4\pi^2\frac{x_{12[m}x_{13n]}}{x_{12}^2x_{13}^2x_{23}^2} \tag{B.9}$$

Putting together the result in (B.9) together with the symmetric part computed before, and rotating back the result to Minkowskian space-time we finally get:

$$I^M(x_1;x_2,x_3) \;=\; -\mathrm{i}4\pi^2\sigma^\mu_{\alpha\dot\alpha}\sigma^\nu_{\beta\dot\beta}\epsilon^{\dot\alpha\dot\beta}\,\frac{x_{12\mu}x_{13\nu}}{x_{12}^2x_{13}^2x_{23}^2} \tag{B.10}$$

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